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A NINE POINT THEOREM FOR 3-CONNECTED GRAPHS

D. A. HOLTON,
B. D. McKAY, M. D. PLUMMER[1],
and
C. THOMASSEN

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We prove that a 3-connected cubic graph contains a cycle through any nine points.

1. Introduction

As proved by Dirac [6] it is an immediate consequence of Menger's theorem that a k-connected graph contains a cycle through any k prescribed points. Watkins and Mesner [12] characterized the graphs which show this to be best possible. These are precisely those k-connected graphs which contain k points whose deletion results in a graph with more than k components. Bondy and Lovász [2] proved that the set of cycles through k specified points in a $(k+1)$-connected graph generates the cycle space of the graph and Holton [8] has shown that the above mentioned result of Dirac can be strengthened for k-connected k-regular graphs.

The line analogue of Dirac's result has also been studied. Lovász [9] conjectured that a k-connected graph has a cycle through any k prescribed independent lines unless these lines form an odd cutset and he verified this for $k=3$. Häggvist and Thomassen [7] proved a weaker conjecture made by Woodall [13] that a $(k+1)$-connected graph has a cycle through any k prescribed independent lines.

In this paper we prove that a 3-connected cubic graph contains a cycle through any nine prescribed points.

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2. Notation and auxiliary results

We shall make use of the following three reductions performed on a cubic 3-connected graph G.

If $e=xy$ is a line of G and $\{x_1, x_2, y\}$ and $\{y_1, y_2, x\}$ are the sets of neighbors of x and y respectively, then the *e-reduction* of G is the multigraph $(G-\{x, y\}) \cup \cup \{x_1x_2, y_1y_2\}$. We note that an e-reduction of a cyclically 4-line-connected cubic graph is 3-connected.

If E is a cutset (i.e., $G-E$ is disconnected) consisting of three independent lines, then a graph obtained from G by contracting a component of $G-E$ to a point is called a *3-cut-reduction* of G. Note that any 3-cut-reduction is cubic and 3-connected. Such a 3-cut-reduction can not yield a multigraph. Further, if G is bipartite, then so is any 3-cut-reduction of G.

Finally, if $C=x_1x_2x_3x_4x_1$ is a 4-cycle in G and the neighbors y_1, y_2, y_3, y_4 of x_1, x_2, x_3, x_4 respectively which are not in C are all distinct, then the cubic multigraphs $G_1=(G-\{x_1, x_2, x_3, x_4\}) \cup \{y_1y_2, y_3y_4\}$ and $G_2=(G-\{x_1, x_2, x_3, x_4\}) \cup \cup \{y_2y_3, y_1y_4\}$ are called *4-cycle-reductions* of G with respect to C. Note that if G is not only cubic and 3-connected, but also cyclically 4-line-connected, then the y_i's must be distinct and hence both 4-cycle-reductions of G with respect to C must exist.

Lemma 1. *If G is a cyclically 4-line-connected cubic graph of order greater than 8 and $C=x_1x_2x_3x_4x_1$ is a 4-cycle of G, then at least one of the 4-cycle-reductions G_1 and G_2 with respect to C is 3-connected.*

Proof. Suppose G_1 is not 3-connected. Then G_1 has a set E_1 of one or two lines such that G_1-E_1 has two components G_1' and G_2'', say. Since G is 3-connected we may assume without loss of generality that y_1y_2 is a line of G_1' and y_3y_4 is a line of G_1''. Further, since G is cyclically 4-line-connected, then $|E_1|=2$.

Similarly we can assume that G_2 contains a set of two lines such that G_2-E_2 has two components G_2' and G_2'' with y_2y_3 in G_2' and y_1y_4 in G_2''. Thus $(G-V(C))-(E_1 \cup E_2)$ has at least four components H_1, H_2, H_3, H_4. Since G is 3-connected at least three lines of G join $V(H_i)$ to $V(G)-V(H_i)$ for each $i=1, 2, 3, 4$. Then a straightforward counting argument implies that there are precisely three lines joining each $V(H_i)$ to $V(G)-V(H_i)$ and, since G is cyclically 4-line-connected, each $V(H_i)$ has cardinality one. Hence the order of G is 8, a contradiction. ∎

Lemma 2. *If G is a 3-connected cubic bipartite graph of order at most 18, then G is Hamiltonian. Furthermore, if G has order at most 14, then any line of G is contained in a Hamiltonian cycle and if G has order at most 10, then any three lines of G are contained in a Hamiltonian cycle unless they form a cutset.*

Proof. The only 3-connected cubic bipartite graphs of order at most 10 are the graphs labelled 6.2, 8.1, 10.13 and 10.16 in [1]. The lemma is easily verified for these graphs.

In Figure 1 we display the cubic bipartite graphs of girth greater than 4 and order at most 18. These graphs satisfy the lemma as well.

We now proceed by induction on $|V(G)|$. If G is not cyclically 4-line-connected, then it must have a cutset consisting of three independent lines. The cor-

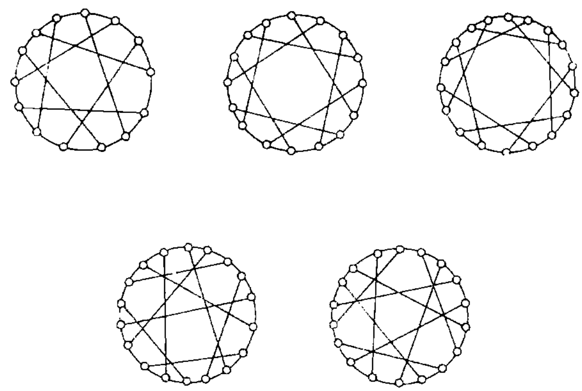

Fig. 1. The cubic bipartite graphs of girth 6 and order at most 18

responding 3-cut-reductions are both 3-connected cubic bipartite simple graphs and we apply the induction hypothesis to each of them. Hence our original graph satisfies the lemma.

So we may assume finally that G is cyclically 4-line-connected and has a 4-cycle. Let G_1 be a 3-connected 4-cycle-reduction of G. (Such must exist by Lemma 1.) Hence by the induction hypothesis, G_1 satisfies the conclusion of the lemma.

Suppose $|V(G)|\leqq 18$. Then $|V(G_1)|\leqq 14$ and by the induction hypothesis there exists a Hamiltonian cycle H_1 in G_1 containing line y_1y_2. But then H_1 easily extends to an H-cycle of G.

Next suppose $12\leqq|V(G)|\leqq 14$. Then $|V(G_1)|\leqq 10$. By the induction hypothesis any three lines of G_1 are in a Hamiltonian cycle of G_1 unless they form a cut set. This easily implies that, for any two distinct lines e_1 and e_2 of G_1, G_1-e_2 has a Hamiltonian cycle containing e_1.

If e_0 is incident with one or two of $\{x_1, x_2, x_3, x_4\}$ then by letting e_1 and e_2 be the two lines of $E(G_1)-E(G)$ the result follows. On the other hand, if e_0 is in $E(G_1)\cap E(G)$ then the result follows by induction. ∎

By inspection of the list of cubic graphs of order at most 10 (see [1]), we also obtain the following result.

Lemma 3. *Let e_1, e_2 be lines of a cubic 3-connected graph G of order at most 10 and let x be a point not incident with e_1 or e_2. Then G has a cycle containing e_1 and e_2 and all points of G except possibly x, unless G is the Petersen graph.*

3. Cycles through specified points in cubic graphs

We now prove the main result of this paper.

Theorem 1. *Let G be a 3-connected cubic graph and A be any set of at most nine points of G. Then G has a cycle containing A.*

Proof. We proceed by induction on $n=|V(G)|$. The Petersen graph is the only non-Hamiltonian 3-connected cubic graph of order at most 10 and it satisfies the theorem, so we may assume we have $n \geqq 12$.

Suppose first that G is not cyclically 4-line-connected. Let G_1 and G_2 be the two 3-cut-reductions of G which correspond to some cutset of three independent lines in G. We choose this cutset to minimize $|A \cap V(G_1)|$ and then among all these to minimize $|V(G_1)|$. If $|A \cap V(G_1)| \leqq 1$, then we obtain the desired cycle by considering a cycle of G_2 containing $A \cap V(G_2)$ together with the point of $V(G_2) - V(G)$ if necessary. Hence, we may assume that $2 \leqq |A \cap V(G_1)| \leqq |A \cap V(G_2)|$. In particular, then $|A \cap V(G_1)| \leqq 4$.

If $|V(G_1)| \leqq 10$, by induction we have a cycle of G_2 containing $A \cap V(G_2)$ and the point in $V(G_2) - V(G)$. By Lemma 3 and a check of the Petersen graph we can extend this cycle to a cycle of G containing A. We may therefore assume that $|V(G_1)| \geqq 12$. Since $|A \cap V(G_1)| \leqq 4$ there is a line e of G_1 which is not incident with the point in $V(G_1) - V(G)$ nor with any point of A. The minimality of $|V(G_1)|$ implies that the e-reduction G_e of G is 3-line-connected and hence 3-connected (since G is cubic), so we can apply the induction hypothesis to G_e.

We can then assume that G is cyclically 4-line-connected. If G contains a line e which is not incident with any point of A, then we apply the induction hypothesis to the e-reduction of G as above. So assume $G - A$ consists of isolated points. In particular, $n \leqq 2|A| \leqq 18$. If $n = 18$, G is bipartite in which case we apply Lemma 2. In fact, via Lemma 2, we may assume that G is not bipartite for $n \leqq 18$.

So assume that $12 \leqq n \leqq 16$, that $G - A$ consists of isolated points and that G is not bipartite.

If G has a 4-cycle, then we let G_1 be a 3-connected 4-cycle-reduction of G (which must exist by Lemma 1). For G_1 not cyclically 4-line-connected we apply the induction hypothesis (combined with the Hamiltonian properties of small cubic graphs) to the two 3-cut-reductions of G_1 associated with some cutset of three independent lines. So we may assume G_1 is cyclically 4-line-connected. Denote by e_0 one of the lines of $E(G_1) - E(G)$. Now let e be a line of G_1 having an endpoint in common with e_0 and we let G_2 be the e-reduction of G_1.

The proof is now easily completed using Lemma 3 and if G_2 is the Petersen graph by doing some case checking which, fortunately, due to the symmetry of the Petersen graph, is easily carried out.

So we know that $12 \leqq n \leqq 16$, $G - A$ is an independent set of points, G is not a bipartite graph, G contains no 3- or 4-cycles (i.e., girth $G \geqq 5$), G is cyclically 4-line-connected and $|A| \leqq 9$. We claim under these conditions that there are two adjacent points of A each joined to $G - A$ by lines e_1 and e_2, respectively.

Suppose not. Then let A_1 be those points of A with neighbours in $G - A$ and let $A_2 = A - A_1$. Further, let $a_i = |A_i|$, for $i = 1, 2$, and let $b = |V(G) - A|$. Now by our supposition, A_1 is independent, so, since G is not bipartite, there must be a line e in A_2 and hence $a_2 \geqq 2$. Then since G is cyclically 4-line-connected there are $\geqq 4$ lines between A_2 and A_1. Now there are $3a_1$ lines leaving A_1 altogether and $3b$ of these go to $G - A$. Hence $3(a_1 - b)$ go to A_2. But $3(a_1 - b)$ is a multiple of 3 and hence $\geqq 6$ lines join A_1 and A_2. But then also $a_1 - b \geqq 2$; i.e., $a_1 \geqq b + 2$.

Now $a_2 \geqq 3$. If $a_2 = 3$ there are two possibilities. Let u denote the point in A_2 which is not an endpoint of e. First, if u sends 3 lines to A_1 we have a total of 7 lines joining A_1 and A_2, contradicting the fact that this number must be multiple of 3. So suppose u sends a line to another point of A_2. Then we have 5 lines joining A_1 and A_2 which again is not a multiple of 3.

Thus we conclude that $a_2 \geqq 4$. Hence since $|A| \leqq 9$, $a_1 \leqq 5$. Also since $V(G) \geqq 12$, $b \geqq 3$. But $a_1 \leqq 5$ and $b \geqq 3$ implies there is a 4-cycle between A_1 and $G-A$, a contradiction and the claim is proved.

(To see that a 4-cycle must exist if $a_1 \leqq 5$ and $b \geqq 3$, simply take any 3 points in $G-A$ and observe where the 9 lines incident with them must have their end-points in A_1.)

So there exist adjacent points x_1 and x_2 in A_2 joined to $G-A$ by lines e_1 and e_2 respectively.

We now consider the e_1-reduction, G_1, of G and the e_2-reduction, G_2, of G_1.

Claim 1. *Since G has girth $\geqq 5$ and G_1 has order $\leqq 14$, G_1 is cyclically-4-line-connected unless G_1 is the 14-point graph shown in Figure 2 below.*

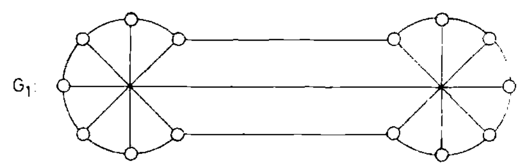

Fig. 2

To prove this claim, suppose G_1 is not cyclically 4-line-connected. Then G_1 has the appearance of Figure 3.

So without loss of generality we may suppose H_1 contains $\leqq 7$ points. Contract H_2 to a point and we obtain a cubic graph G_1^* with $\leqq 8$ points.

Now remember G must have the appearance of Figure 4 where the line e_1 joins two points marked "x" one on a line in H_1, the other on a line in H_2 and that G has girth $\geqq 5$.

There are 8 possibilities for G_1^* (see [1]). In particular, since girth $G \geqq 5$, girth $G-e_1 \geqq 5$ and hence subdividing a single line of G_1^* must destroy all cycles of length $\leqq 4$ in H_1. Thus G_1^* can only be graph 8.2 of [1] and G_1 has the appearance of Figure 5.

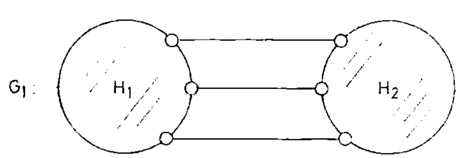

Fig. 3

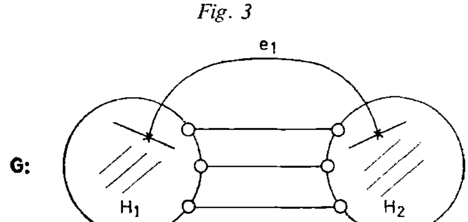

Fig. 4

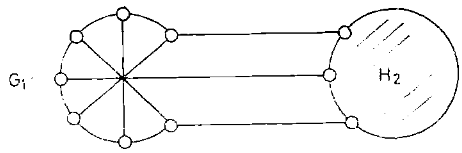

Fig. 5

But now H_2 has $\leqq 7$ points so by a completely symmetric argument, we have G_1 as shown in Figure 2 and Claim 1 is proved.

Now if G_1 is not cyclically 4-line-connected, G_1 must be the graph of Figure 2 by Claim 1. Moreover. G must then be the graph of Figure 6.

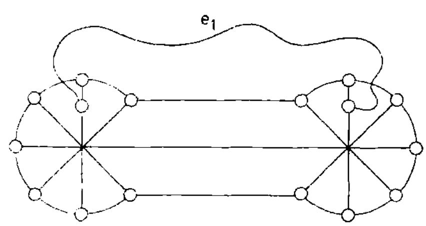

Fig. 6

But this graph is easily seen to have a Hamiltonian cycle and we are done in this case.

So we may assume in the following that G_1 is cyclically 4-line-connected. But it then follows that the e_2-reduction G_2 of G_1 is 3-connected. Moreover, G_2 has $\leqq 12$ points.

Recall that we know that G contains two adjacent points x_1, x_2 of A joined to points y_1, y_2 of $G-A$ by lines e_1, e_2 respectively.

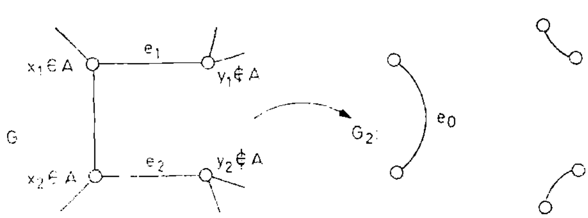

Fig. 7

Let e_0 denote the line in G_2 "containing" points x_1 and x_2 of G (see Figure 7). A proof of the following Claim suffices to complete the proof of our theorem.

Claim 2. *If G_2 is 3-connected and cubic with $\leqq 12$ points, if e_0 is any line of G_2 and if $|A \cap V(G_2)| \leqq 7$, then G_2 has a cycle through e_0 and $A \cap V(G_2)$.*

The proof of this Claim will be by induction on $|V(G_2)|$ and we shall treat two cases.

Case 1. Suppose G_2 is not cyclically 4-line-connected. Then G_2 has the appearance of Figure 8 where we may assume without loss of generality that e_0 is in H_2. Now we "push" the 3-cut "to the left" as far as possible in the sense that with e_0 in H_2 we choose the 3-cut so that $|V(H_1)|$ is as small as possible.

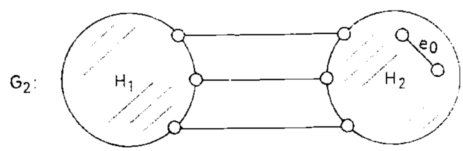

Fig. 8

Now contract H_1 to a single point u and call the resulting graph J. Then $|V(J)| \leqq 10$.

If $|V(J)| = 10$, then G_2 has the appearance of Figure 9.

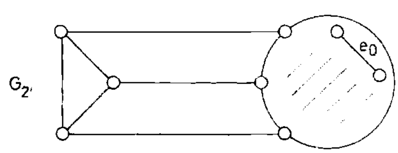

Fig. 9

Then the induction hypothesis applies to J and a cycle of the desired type exists in J. (Note here we adopt the convention that $u \in V(J) \cap A$ iff $V(H_1) \cap A \neq \emptyset$.) Moreover, this cycle extends to a cycle in G_2 containing e_0 and $A \cap V(G_2)$.

So $|V(J)| \leqq 8$. But checking the 3-connected cubic graphs with $\leqq 8$ points (again see 4.1, 6.1, 6.3, 8.1, 8.2, 8.3, 8.5, of [1]) we see that J must contain a Hamiltonian cycle containing e_0. This cycle corresponds to a path R containing two of the lines of the 3-cut in G_2 and containing a Hamiltonian path in H_2 which in turn contains e_0. (See Figure 10.)

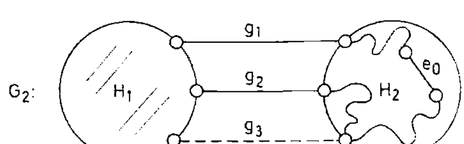

Fig. 10

Now contract H_2 to a point v and call the resulting graph on $\leqq 10$ points F. If F has $\leqq 8$ points, by checking [1] yet again, we see that F contains a Hamiltonian cycle S which in turn contains g_1 and g_2. We can then use R and S to get a Hamiltonian cycle for G_2 which contains e_0 and we are done.

So we may assume that F has 10 points. Thus G_2 has the appearance of Figure 11.

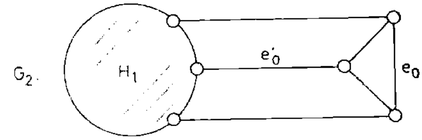

Fig. 11

Let e_0' be that line of the 3-cut not adjacent with line e_0. Then applying the induction hypothesis to F and line e_0' we get a cycle which extends to one in G_2 containing e_0 and all of $V(G_2)\cap A$. This completes Case 1.

Case 2. Assume G_2 is cyclically 4-line-connected. (Recall this implies in turn that G_2 has no cut sets of 3 independent lines and no triangles.)

If G_2 has $\leqq 10$ points an appeal to [1] (more precisely to 8.1, 8.2, 10.13, 10.14, 10.15, 10.17 and 10.19) shows that all have cycles through any preassigned line and 7 (or fewer) preassigned points.

So we may assume that G_2 has 12 points. Let f_1 and f_2 be two lines adjacent to the same end of e_0 and let z_1 and z_2 be the endpoints of f_1 and f_2 respectively which are not on e_0. (Cf. Figure 12.)

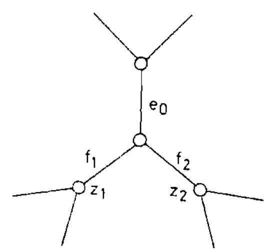

Fig. 12

Suppose one of z_1 and z_2, say z_1, is not in A. Then apply the induction hypothesis to the f_1-reduction of G_2 to get a cycle through e_0 and $A\cap V(G_2)$ which extends to a cycle in G_2 with these properties.

So suppose both z_1 and $z_2 \in A$. Let G_3 denote the f_1-reduction of G_2 (cf. Figure 13).

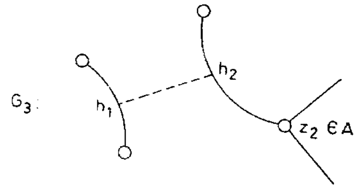

Fig. 13

Now G_3 has 10 points. Assume for the moment that G_3 is not the Petersen graph. We want to apply Lemma 3 to G_3, but in order to do that we need a candidate for the point "x" in that Lemma. Let h_1 and h_2 denote the two new lines formed in getting G_3, the f_1-reduction of G_2. There are at most 5 points of A in G_3 which are not endpoints of h_1 and h_2. But G_3 has 10 points and hence there is a point x in G_3 which is neither in A nor an endpoint of h_1 or h_2.

Now we can apply Lemma 3 to conclude that G_3 has a cycle through h_1, h_2 and all 5 other points of A in G_3. But of course this cycle extends to one in G_2 containing all 7 points of $A \cap V(G_2)$ as well as e_0 completing Case 2 and Claim 2, unless G_3 is the Petersen graph. But if G_2 arises from the Petersen graph by inserting a new line joining the midpoint of some two independent lines, it is routine to show the even stronger result that such a G_2 will in fact have a Hamiltonian cycle through any preassigned line. Claim 2 is thus proved and the proof of the theorem is complete. ∎

Theorem 1 is best possible as demonstrated by any graph contractible to the Petersen graph. We believe that every extremal graph has this property.

4. Concluding remarks

The last part of the proof of Theorem 1 would follow from the result that any cyclically 4-line-connected cubic graph with at most 16 points is Hamiltonian or isomorphic to the Petersen graph. This conjecture is certainly true for $n \leqq 14$ as proved in [4]. Such a result would be best possible as there are hypohamiltonian cubic graphs of every even order greater than 16 (see [5, 11]).

Lemma 2 is probably far from best possible. The smallest known 3-connected cubic bipartite non-Hamiltonian graph is that of Horton [3, page 240] which has order 96.

Define $f(k)$ to be the largest positive integer such that any k-connected k-regular graph has a cycle through any specified set of at most $f(k)$ points. We have proved in this paper that $f(3)=9$. Watkins and Mesner's result [12] shows that $f(k) \geqq k+1$ and Holton [8] proved that $f(k) \geqq k+4$. Meredith [10] gave examples of k-connected k-regular non-Hamiltonian graphs for each $k \geqq 3$. These are obtained by replacing each point of the Petersen graph by a copy of $K(k, k-1)$. In particular, since there is no cycle through all sets of size $k-1$ in the bipartition of each replacement $K(k, k-1)$, we have proved $f(k) \leqq 10k-11$.

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D. A. Holton

University of Melbourne
Melbourne, Australia

B. D. McKay

Dept. Computer Science
Vanderbilt University
Nashville, Tenn. 37235
U.S.A.

M. D. Plummer

Dept. Mathematics
Vanderbilt University
Nashville, Tenn. 37235
U.S.A.

C. Thomassen

Matematisk Institut
Bygn. 303
Technical University of Denmark
2800 Lyngby
Denmark